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Minimal Peano Curve

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Abstract—A Peano curve $p(x)$ with maximum square-to-linear ratio $\frac{|p(x)-p(y)|^2}{|x-y|}$ equal to $5\frac{2}{3}$ is constructed; this ratio is smaller than that of the classical Peano–Hilbert curve, whose maximum square-to-linear ratio is 6. The curve constructed is of fractal genus 9 (i.e., it is decomposed into nine fragments that are similar to the whole curve) and of diagonal type (i.e., it intersects a square starting from one corner and ending at the opposite corner). It is proved that this curve is a unique (up to isometry) regular diagonal Peano curve of fractal genus 9 whose maximum square-to-linear ratio is less than 6. A theory is developed that allows one to find the maximum square-to-linear ratio of a regular Peano curve on the basis of computer calculations.

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1. INTRODUCTION

Continuous mappings of an interval onto a square, called Peano curves in honor of the Italian mathematician Giuseppe Peano, who discovered these curves, turned from a mathematical oddity into a tool of applied mathematics a long time ago.

Peano curves are used for numerically integrating functions of several variables, for compressing images, and for encoding information (see [1, 3]). A two-dimensional image (black and white, grayscale, or color) can be represented as a function $f(x, y)$ defined on a (digital) rectangle. Let $p(t)$ be a Peano curve that maps an interval onto this rectangle. Then the composition $f(p(t))$ is a function of one variable, which can be compressed (with loss of information), for example, by decomposing into wavelets. Such a representation agrees well with the JPEG-2000 algorithm and allows for zooming: decoding a part of the image.

An important characteristic of a Peano curve is given by its *square-to-linear ratio*. For a pair of points[1] $p(t)$, $p(\tau)$ of a Peano curve $p\colon [0, 1] \to [0, 1] \times [0, 1]$, the ratio

$$\frac{|p(t) - p(\tau)|^2}{|t - \tau|}$$

is called the *square-to-linear ratio* of the curve p on this pair. The upper bound of the square-to-linear ratios for all possible pairs of different points of a curve is called the square-to-linear ratio of this curve. For applications, the curves with the least square-to-linear ratio are of most interest.

Just as an ordinary curve is naturally parameterized by its length, a Peano curve is naturally parameterized by its area. Namely, a Peano curve $p(t)$ is said to be *area-parameterized* if the area of the image of any interval is equal to the length of this interval. All the Peano curves considered below are assumed by default to be area-parameterized.

According to [2], a mapping of an interval onto a square is called a *regular fractal Peano curve* if the domain of definition can be decomposed into several equal segments (fractal periods) such that

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[1] By a point of a curve we mean a point of its graph; i.e., a point of a Peano curve is in fact a pair $t, p(t)$, where t belongs to the source interval and $p(t)$ belongs to the image square.

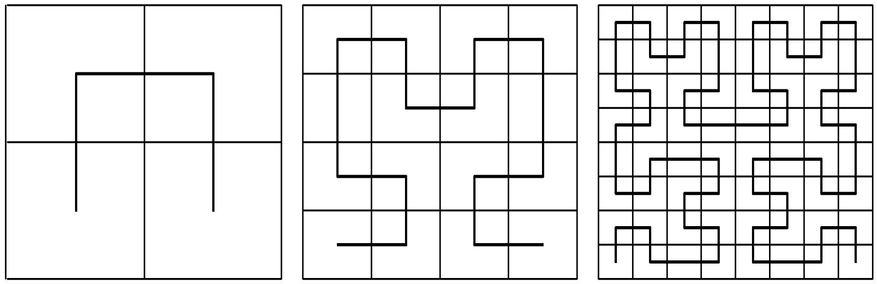

Fig. 1. Three steps of the construction of the Peano–Hilbert curve.

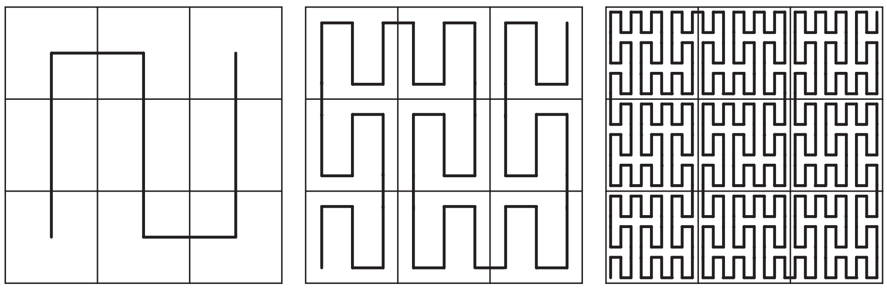

Fig. 2. The first Peano curve.

the restriction of the curve to any of its fractal periods is similar to the whole curve. The (least possible) number of fractal periods is called the *fractal genus* of the curve. The fractal genus of a Peano curve whose image is a square is itself a perfect square. The Peano–Hilbert curve is the only (up to symmetry and similarity) curve of fractal genus 4.

As is obvious, a regular fractal Peano curve that maps a unit interval onto a unit square (a unit curve) is area-parameterized.

The exact calculation of the square-to-linear ratio for a given regular Peano curve is a rather difficult problem. For example, until the present study, the square-to-linear ratio was exactly determined (see [4]) only for the simplest (and the most popular) Peano–Hilbert curve (see Fig. 1), and this ratio is 6.

On the other hand, Shchepin [2] proved that the square-to-linear ratio of any unit regular Peano curve cannot be less than 5.

The Peano–Hilbert curve has fractal genus 4; i.e., it is divided into four isometric parts that are similar to the whole curve. The original Peano curve, which is shown in Fig. 2, has fractal genus 9. The square-to-linear ratio for this curve is 8.

In this paper, we systematically analyze Peano curves of fractal genus 9. The number of such curves amounts to thousands. Among them, there is (Theorem 1) a unique curve (see Fig. 3) with square-to-linear ratio $5\frac{2}{3}$, which is smaller than that of the Peano–Hilbert curve. We will call this curve a *minimal* N-*shaped* curve.

The proof of the fact that the square-to-linear ratio of the minimal N-shaped curve is equal to $5\frac{2}{3}$ is based on computer calculations. To guarantee the validity of a proof based on these calculations, in this paper we construct a theory that allows one to draw exact conclusions about the square-to-linear ratio on the basis of computer calculations.

The restriction of a curve onto its fractal period is called a *fraction* of this curve. A regular Peano curve of fractal genus g is divided into g isometric fractions of the first order. In turn, each first-order fraction is divided into g isometric fractions of the second order, etc. All kth-order fractions are isometric to each other and are similar to the whole curve. In this paper, we prove (Theorem 3) that for "nonsingular" (without singular points in the sense of Section 4) regular Peano

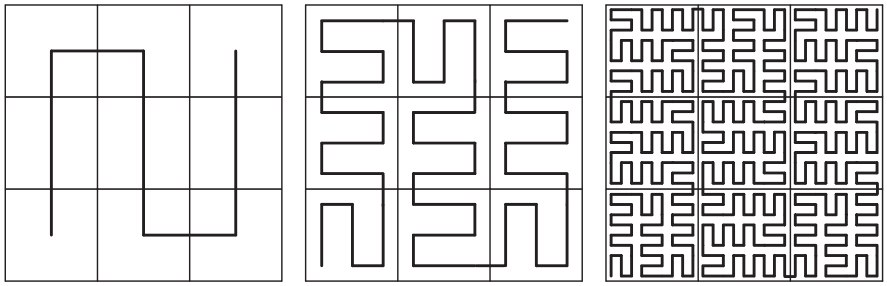

Fig. 3. Minimal N-shaped curve.

curves, the square-to-linear ratio attains its maximum on a pair of corners (the image of a fraction is a square) of fractions of some order. Nonsingular curves include, in particular, all diagonal curves whose square-to-linear ratio is less than 6 (Theorem 2).

The key concept that defines the volume of computations necessary for the validity of the proof is the concept of *depth* of a regular Peano curve. The restriction of a Peano curve to a pair of adjacent fractal periods of order k is called a kth-order *junction* of this curve. The depth of a regular fractal Peano curve is the greatest natural number d for which the curve has a junction of order d that is not similar to any junction of lower order. The depth of the Peano–Hilbert curve is equal to 2, while the depth of the minimal N-shaped curve is 1. It can be shown that the depth of any regular Peano curve is not greater than 9.

As a rule, for a regular Peano curve of depth d (see Theorem 7), the square-to-linear ratio attains its maximum on a pair of points that are corners of fractions of order $d+4$. In particular, this is the case for the minimal N-shaped curve.

2. RECURRENT EQUATIONS OF SQUARE PEANO CURVES

Chain code. We consider a plane equipped with a complex structure. We assume that the real axis is horizontal and is directed from left to right, while the imaginary axis is vertical and is directed upward.

A sequence of squares with the sides parallel to the coordinate axes of a plane square lattice is called a *chain* if any two adjacent squares in the sequence have a nonempty intersection. The *caliber* of a chain is the (common) side length of the squares contained in this chain. The difference between the center of a square of a chain and the center of the preceding square is called a *step* of the chain. A *normalized step* of a chain is the ratio of a step to the caliber. The normalized step may take one of the eight values: 1, i, -1, $-i$, $1+i$, $1-i$, $-1+i$, or $-1-i$. A chain of squares is uniquely determined by the position of the first square and the sequence of normalized steps. The polygonal line connecting the centers of adjacent squares of a chain is called the *central polygonal line* of the chain of squares. The central polygonal line uniquely determines the chain and is used for its visualization.

The union of all squares of a chain is called the *body* of the chain. A chain of squares is called a *subdivision of another*, with the larger caliber, if these chains have identical bodies and the finer chain is divided into nonintersecting chains whose bodies make up the squares of the coarser chain. A sequence of chains in which each next chain is a subdivision of the preceding is called a *refining* sequence. Any refining sequence of chains of squares defines a continuous mapping of a unit interval of the real axis onto the common body of the chains of the sequence. This mapping can be defined as the limit of naturally (by the arc length) parameterized central polygonal lines of the chains of the sequence.

To code the central polygonal lines of chains, we will use special sums enclosed in brackets. The terms of such a sum are the normalized steps of the chain, and their order corresponds to the order of the squares in the chain. We will call these sums *chain codes*. For example, the expression

$$[i + 1 - i] \tag{1}$$

corresponds to the polygonal line whose second square is situated above the first square, the third square is to the right of the second, and the fourth is below the third. Such is the central polygonal line of the first subdivision of the Peano–Hilbert curve (see Fig. 1). The central polygonal line of the second subdivision of the same curve is expressed as

$$\big[1 + i - 1 + i + i + 1 - i + 1 + i + 1 - i - i - 1 - i + 1\big]. \tag{2}$$

We consider the symbols of a chain code as complex numbers. Therefore, we can apply the operation of termwise multiplication by a complex number to a chain code. For example, $i[i+1-i] = [-1+i+1]$. Complex conjugation is also applied to chain codes termwise: $\overline{[-i+1+i]} = [i+1-i]$. Using this notation and denoting the code $[i+1-i]$ by a single letter d, we can represent the above code of the second subdivision as

$$\big[i\bar{d} + i + d + 1 + d - i - i\bar{d}\big]. \tag{3}$$

This formula is universal; it allows one to obtain the code of the $(n+1)$th subdivision by substituting the code of the nth subdivision for d. Thus, we have obtained a *recurrent equation* of the central polygonal line for the Peano–Hilbert curve:

$$d_{n+1} = \big[i\bar{d}_n + i + d_n + 1 + d_n - i - i\bar{d}_n\big], \qquad d_0 = [\,]. \tag{4}$$

The recurrent equation that describes the polygonal lines passing through the corners of the fractions of the subdivisions of this curve is still simpler. For the initial square, the order of corners in which they are passed by the curve is defined by the same chain code (1), and the recurrent equation of the corner polygonal line is given by

$$d_{n+1} = \big[i\bar{d}_n + d_n + d_n - i\bar{d}_n\big], \qquad d_0 = [i + 1 - i]. \tag{5}$$

Bauman [4] proved that the square-to-linear ratio of the Peano–Hilbert curve is equal to 6.

The first Peano curve. The Peano–Hilbert curve analyzed above was constructed by Hilbert. However, the first curve that sweeps out a square was constructed by Peano himself and is more complicated. As will be shown below, it has a much greater square-to-linear ratio than the Peano–Hilbert curve. Therefore, the first Peano curve is little known. The first three approximations to this Peano curve are shown in Fig. 2.

The first Peano curve satisfies the following symmetry condition: $p(1-t) = 1 + i - p(t)$.

The beginning and the end of the Peano–Hilbert curve are situated on the same side of the image square. We call the curves possessing such a property *one-sided* curves. The beginning and the end of the first Peano curve lie at opposite vertices of the image square. The curves possessing this property are said to be *diagonal*.

The recurrent equation of the central polygonal lines of the first Peano curve is as follows:

$$d_{n+1} = \big[d_n + i - \bar{d}_n + i + d_n + 1 + \bar{d}_n - i - d_n - i + \bar{d}_n + 1 + d_n + i - \bar{d}_n + i + d_n\big], \qquad d_0 = [\,].$$

Corner moments of the first Peano curve. To describe Peano curves, we apply the language of motion, so that the interval to be mapped is interpreted as a time interval, and its points are called moments.

Note that for a corner of any fraction of a regular subdivision of a square, there exists a unique fraction of the next subdivision that contains this corner. Therefore, the moment when the curve

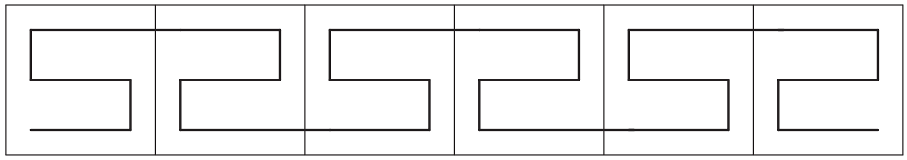

Fig. 4.

passes through a corner of a given fraction is uniquely defined in the fractal period corresponding to this fraction. Different fractions may have common corners; therefore, the moment of a corner of a fraction is correctly (uniquely) defined provided that it is specified to which fraction the corner belongs.

Since the third fraction (of the first subdivision) of the first Peano curve is similar to the whole curve, the difference between the moments of the upper left corner of the third fraction and the beginning of the third fraction is 9 times smaller than the moment of the upper left corner of the image square. Since the upper left corner of the image square coincides with the upper left corner of the third fraction and the starting moment of the third fraction is obviously equal to $\frac{2}{9}$, we obtain the following equation for the first corner moment:

$$x = \frac{2}{9} + \frac{x}{9}, \tag{6}$$

which implies that the moment of the upper left corner of the first Peano curve is equal to $\frac{1}{4}$. Similar arguments (together with symmetry considerations) allow one to determine the moment of the lower right corner as $\frac{3}{4}$.

The rectangle composed of the fractions $16, 17, \ldots, 21$ of the second subdivision (with a side $\frac{1}{9}$) and rotated clockwise through an angle of $90°$ is shown in Fig. 4.

The difference between the moments of the upper left and the upper right corners of this rectangle is $\frac{\frac{1}{4}+4+\frac{1}{4}}{81}$, while the distance between these corners is $\frac{2}{3}$. As a result, the square-to-linear ratio on this pair of points is 8. Thus, the square-to-linear ratio of the first Peano curve is at least 8.

Minimal N-shaped curve. There are thousands of regular (in the sense of [2]) Peano curves of fractal genus 9 with the chain code of the first subdivision equal to $[i + i + 1 - i - i + 1 + i + i]$, i.e., to the chain code of the first subdivision of the first Peano curve. We call all these curves N-shaped because of the visual similarity to this letter. Computations performed by the authors show that the curve presented in Fig. 3 has the minimal square-to-linear ratio among all regular Peano curves of genus 9.

The recurrent equation of the central polygonal lines of the minimal N-shaped curve is as follows:

$$d_{n+1} = \big[d_n + i + id_n + i + i\bar{d}_n + 1 + \bar{d}_n - i - i\bar{d}_n - i - id_n + 1 + d_n + i + id_n + i + i\bar{d}_n\big]. \tag{7}$$

Corner moments of the minimal curve. Denote the moment of the corner B by s and the moment of the corner C by t (Fig. 5).

Obviously, the moment of the point A' is equal to $\frac{2}{9}$. The difference of moments between A' and B is equal to $\frac{1}{9}$ of the difference of moments between A and C in view of the similarity between the third fraction and the whole curve. As a result, we obtain the equation $s = \frac{2}{9} + \frac{t}{9}$.

Similarly, the difference of moments between C and D is formed by the interval from D to D', equal to $\frac{2}{9}$, and the interval from C to D', which, in view of similarity, is equal to $\frac{1-t}{9}$; i.e., $1 - t = \frac{2}{9} + \frac{1-t}{9}$. Thus, we obtain the following system of equations for the corner moments:

$$s = \frac{2}{9} + \frac{t}{9}, \qquad 1 - t = \frac{2}{9} + \frac{1-t}{9}; \tag{8}$$

solving this system, we find $s = \frac{11}{36}$ and $t = \frac{3}{4}$.

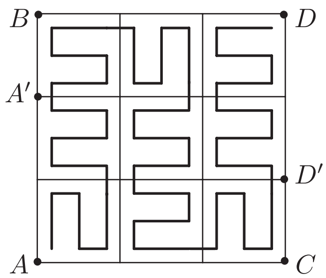

Fig. 5.

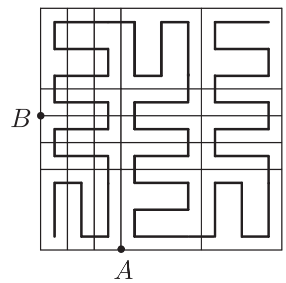

Fig. 6.

Knowing the corner moments, we can determine the square-to-linear ratio for the pair of points A, B shown in Fig. 6.

The moment of the point A is the second corner moment of the first fraction and is therefore equal to $\frac{3}{4} \times \frac{1}{9} = \frac{1}{12}$. The point B is the second (according to the traversal order) corner of the 13th second-order fraction. Therefore, its moment is equal to $\frac{12}{81} + \frac{3}{4} \times \frac{1}{81} = \frac{17}{108}$. Hence, the difference of moments between A and B is equal to $\frac{2}{27}$. Since the squared distance is equal to $\frac{5^2}{9^2} + \frac{3^2}{9^2} = \frac{34}{81}$, the square-to-linear ratio for this pair is $\frac{34}{81} : \frac{2}{27} = \frac{34}{6} = 5\frac{2}{3}$. Below, we will show that the maximum of the square-to-linear ratio of the minimal N-shaped curve is equal precisely to $5\frac{2}{3}$.

3. UNIQUENESS THEOREM

Theorem 1. *There exists a unique, up to isometry, regular diagonal Peano curve of fractal genus* 9 *that maps a unit interval onto a unit square and whose square-to-linear ratio is less than* 6. *This curve is defined by equation* (7).

The proof of Theorem 1, which is the subject of this section, is divided into a series of lemmas.

To clarify the situation, we will use the letters N, И, S, and Z to denote the fractions of an N-shaped curve that have the following chain codes of the first subdivision: $\mathrm{N} = [i+i+1-i-i+1+i+i]$, $\mathrm{И} = \overline{\mathrm{N}} = [-i-i+1+i+i+1-i-i]$, $\mathrm{S} = i \cdot \overline{\mathrm{N}}$, and $\mathrm{Z} = -i \cdot \mathrm{N}$.

Thus, for N, the curve starts moving upward from the lower left corner; for И, the curve starts at the upper left corner and moves downward; for S, the curve starts moving from the lower left corner to the right; and for Z, from the lower right corner to the left. In this case, the second subdivision of an N-shaped Peano curve can be represented as a 3×3 matrix.

The order of traversal of fractions gives the nine-letter code NИNИNИNИN for the first Peano curve and NZSИSZNZS for the minimal curve. We call the codes N and И vertical and the codes Z and S horizontal. A pair of adjacent symbols of a nine-letter code is called vertical (horizontal) if the corresponding elements of the nine-element matrix lie in the same column (row).

Lemma 1. *In the nine-letter code of any continuous* N*-shaped curve,*

(1) *two identical symbols cannot be consecutive*;

(2) *the symbols* N *and* S *cannot be consecutive*;

(3) *the symbols* И *and* Z *cannot be consecutive.*

Proof. Each square in the first subdivision chain has its beginning and end on the diagonal. The beginning of the next square in the chain must coincide with the end of the preceding square. These arguments allow us to prove that the combinations listed above are impossible. □

Lemma 2. *If there is a vertical pair of vertical symbols or a horizontal pair of horizontal symbols in the nine-letter code of an* N*-shaped curve, then the square-to-linear ratio of this curve is greater than* 6.

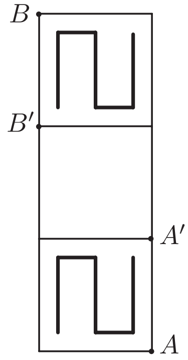

Fig. 7.

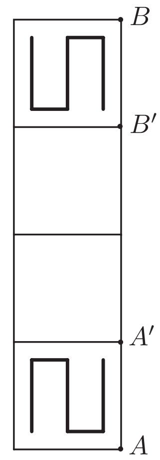

Fig. 8.

Proof. In the cases listed above, the curve contains a straightened sequence of six second-order fractions, as is shown in Fig. 4. Then, the difference of moments between the upper corners is less than $\frac{6}{81} = \frac{2}{27}$ (six times the fractal period), whereas the squared distance between the points is equal to $\frac{4}{9}$. Therefore, the square-to-linear ratio turns out to be greater than $\frac{4}{9} : \frac{2}{27} = 6$. □

Lemma 3. *For an* N*-shaped curve, the first corner moment is not greater than* $\frac{1}{3}$, *while the second cannot be less than* $\frac{2}{3}$.

Proof. Indeed, the first corner is inside the third fraction of the first subdivision, and the second is inside the seventh fraction. □

Lemma 4. *If the square-to-linear ratio of an* N*-shaped curve is less than* 6, *then the matrix of its second subdivision cannot contain a vertical triple of consecutive symbols two of which are vertical.*

Proof. Consider the case when the top and bottom fractions are N and N. This case is illustrated in Fig. 7. Then the curve passes the segments AA' and BB' in time no greater than $\frac{1}{3}$ of the fractal period, which is equal to the time between A' and B'. Thus, the curve passes AB in time no greater than $\frac{5}{3} : 9 = \frac{5}{27}$, and the squared distance is equal to $\frac{10}{9}$. As a result, the square-to-linear ratio for the pair A, B is 6. The case of two И is analyzed similarly. The remaining case of two consecutive vertical codes is impossible according to Lemma 2. □

Lemma 5. *If the square-to-linear ratio of an* N*-shaped curve is less than* 6, *then the nine-letter code of its second subdivision is either* NZSИSZNZS *or* SZNZSИSZN.

Proof. If the code starts with N, then the next two symbols must be horizontal by Lemma 4. Therefore, they are uniquely identified as Z and S by Lemma 1. The fourth symbol cannot be horizontal in view of Lemma 2; therefore, it is И. Now, by Lemma 4, the fifth and sixth symbols are horizontal. Therefore, they are uniquely identified as S and Z (Lemma 1). The seventh symbol is vertical by Lemma 2. Therefore, the seventh symbol is N. By Lemma 4, we conclude that the eighth and ninth symbols are horizontal. Therefore, they coincide with Z and S by Lemma 1.

Similarly one can show that a code ending with N has the form SZNZSИSZN. Among the other letters, only S may be the first or the last letter of the code, because the curve begins at the lower left corner and ends at the upper right corner.

To complete the proof of the lemma, we must show that the nine-letter code of an N-shaped curve cannot start and end with S. Suppose the contrary. Then, since any fraction is similar to the whole curve, we find that the nine-letter code of any fraction starts and ends with the same symbol, and the type (vertical or horizontal) of this symbol is different from the type of the symbol of the whole fraction. In this case, any two neighboring fractions cannot be either horizontal or vertical simultaneously. Indeed, if they are both horizontal, then they cannot form a horizontal pair by Lemma 2; however, the junction cannot be vertical either, because otherwise the nine-letter codes of these fractions start and end with vertical symbols, and the last symbol of the first fraction and the first symbol of the second fraction form a vertical pair, contrary to Lemma 2. As a result, the nine-letter code of the curve is uniquely determined as SИSИSИSИS. Then the middle column of the matrix of the second subdivision contains two vertical symbols, which contradicts Lemma 4. □

Now, we proceed to study the structure of the second subdivision of the minimal N-shaped curve. The second subdivision consists of 81 fractions; to code these fractions, we will use the same four-letter alphabet but with lowercase letters instead of capitals.

Lemma 6. *If the square-to-linear ratio of an* N*-shaped curve is less than* 6*, then the matrix of the third subdivision contains neither a vertical column of four elements two of which are vertical nor a horizontal row of four elements with two horizontal codes.*

Proof. We will restrict ourselves to the vertical case. The fractal period of the second order is equal to $\frac{1}{81}$, and the length of a side of a fraction is $\frac{1}{9}$. It takes the curve less than $\frac{1}{3}$ of the fractal period to pass the segments AA' and BB' (see Fig. 8); therefore, the time interval between A and B is less than $2\frac{2}{3} \times \frac{1}{81} = \frac{8}{243}$, and the squared distance for AB is equal to $\frac{16}{81}$. Hence, the square-to-linear ratio for AB is greater than 6. □

A fraction of the first subdivision (coded by a capital letter) is said to be *primitive* if its nine-letter code (consisting of lowercase letters) starts with a letter of the same (horizontal or vertical) type as the type of the whole fraction.

Lemma 7. *If the square-to-linear ratio of an* N*-shaped curve is less than* 6 *and the nine-letter code of its second subdivision is* NZSИSZNZS*, then all fractions of the first subdivision are primitive.*

Proof. Let us first prove that the second fraction is primitive. If the first fraction is primitive, then its nine-letter code ends with nzs arranged in a vertical column. By Lemma 6, this implies that the first symbol of the nine-letter code of the second fraction cannot be vertical. Hence, it is horizontal, like the Z-code of the fraction. If the first fraction is not primitive, then its code ends with a vertical element. The next symbol of the 81-letter code, the one with which the nine-letter code of the second fraction starts, is situated above the end symbol of the code of the first fraction and therefore cannot be vertical in view of Lemma 2. Thus, the second fraction starts with a horizontal code. The primitivity of the fifth and eighth fractions, which are preceded by fractions with code N, is proved analogously.

Now, let us show that the primitivity of the second fraction implies the primitivity of the third fraction. Since the second fraction starts with a horizontal code, it ends with a vertical code. Therefore, the first symbol of the nine-letter code of the third fraction cannot be vertical. Hence, it is horizontal, which proves the primitivity of the third fraction. In the same way, the primitivity of the fifth and eighth fractions imply the primitivity of the sixth and ninth fractions. It remains to establish the primitivity of vertical fractions: the first, fourth, and seventh ones.

Since the third fraction is primitive, the seventh symbol of its code is horizontal (because the third fraction, just as any other primitive fraction, is obtained from a fraction with the code nzsиsznzs by an isometry of the plane). Now, Lemma 6 implies that the code of the fourth fraction starts with a vertical symbol. The primitivity of the seventh fraction is proved analogously.

The primitivity of the first fraction will be proved by contradiction. If the first fraction is not primitive, then the 81-letter code of the curve has the form $[\mathrm{s}\dots\mathrm{n}]$. Consider now the 81-letter code of the third fraction. Since this fraction is S-shaped and primitive, this code has the form $-i\overline{[\mathrm{s}\dots\mathrm{n}]}$ and ends with s, while the 81-letter code of the И-shaped fourth fraction starts with z. Thus, we obtain a horizontal pair of horizontal symbols, which contradicts Lemma 2. □

Chain codes of the form $\pm 1 \pm i$ are said to be *diagonal*.

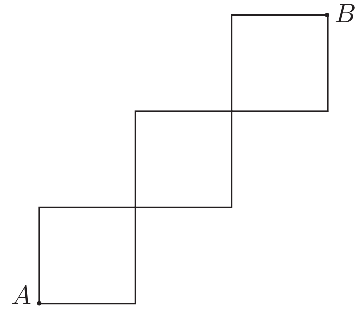

Fig. 9.

Lemma 8. *If the chain code of the central polygonal line of the first subdivision of a diagonal Peano curve* (*of any genus*) *contains a diagonal code, then its square-to-linear ratio is not less than* 6.

Proof. Suppose that the central polygonal line contains two diagonal codes in succession. Then we obtain a triple of consecutive fractions arranged along a diagonal, as in Fig. 9.

Then the square-to-linear ratio between the beginning of the first fraction A and the end of the third B is equal to $(3^2 + 3^2)/3 = 6$.

Thus, in what follows we assume that there are no two consecutive diagonal codes. This fact applies to any subdivision.

Consider an arbitrary diagonal junction PQ of fractions of the first subdivision in more detail; namely, consider the second subdivision. The last fraction of the second subdivision lying in P is linked diagonally to the first subfraction of Q; therefore, the second subfraction cannot continue the diagonal motion and turns to a side. As a result, the next two subfractions of Q (the second and third) lie on a side of Q. Similar arguments show that the two subfractions of P that precede the last subfraction in P lie on a side of P. If we consider in addition the fourth subfraction of Q and the fourth subfraction of P from the end, the set of these eight fractions of the second subdivision can be arranged either in a 5×5 square (fractions of the second subdivision are assumed to have a size of 1×1) or in a 6×4 rectangle. In both cases the sum of squares of the sides is greater than 48; therefore, the square-to-linear ratio is greater than 6. □

Lemma 9. *If the square-to-linear ratio of a diagonal Peano curve of fractal genus* 9 *is less than* 6, *then this curve is isometric to an* N-*shaped curve.*

Proof. Since diagonal junctions are prohibited (by Lemma 8), a curve satisfying the conditions of the lemma must have a chain code of the first subdivision that starts with either i or 1. Furthermore, this curve must have a code ending with either i or 1, because the curve obtained from a given curve by time reversal satisfies the conditions of the lemma.

If the first code is i, then the second code is also i because the beginning of the third fraction coincides with $\frac{2i}{3}$. On the other hand, if the last code is i, then the penultimate code is also i. In this case, it is already obvious that the curve is N-shaped. If the last code is 1, then the penultimate code is also 1.

In this case, the third square of the chain coincides with the eighth square, which is impossible. If the first code is 1, we obtain a curve symmetric to an N-shaped curve. □

Lemma 10. *There exists a unique Peano curve of fractal genus* 9 *whose central polygonal line of the second subdivision has the nine-letter code* NZSИSZNZS *and all of whose fractions are primitive.*

Proof. Let us show that in this situation the functional equation of the curve is defined uniquely. Indeed, to define a functional equation, one should define a similarity between any fraction and the whole curve. If $p(x)$ denotes the whole curve, then the restriction of the curve to the fractal period is isometric to the curve $\frac{p(9x)}{3}$. An isometry of curves is defined by isometries of the domains of definition and images. Since an isometry must map the beginning of a curve to the beginning, there are only two variants of isometry for the images, one of which generates a fraction with a vertical symbol, and the other, a fraction with a horizontal symbol. Since we know the type of the symbol of a fraction, the isometry of the images is defined uniquely. For the isometry of preimages, there are two variants: a time-preserving and a time-reversing ones. However, only the time-preserving variant is compatible with the primitivity of fractions. Thus, the functional equation and hence the curve itself are uniquely defined. □

Proof of Theorem 1. By Lemma 9, any curve satisfying the conditions of the theorem is isometric to an N-shaped curve. By Lemma 5, the nine-letter code of the curve is either NZSИSZNZS or SZNZSИSZN. In the first case, by Lemma 7, all fractions of the curve are primitive. Therefore, in view of Lemma 10, this curve is unique.

In the second case, reversing time, i.e., passing from the curve $p(t)$ to the curve $p(1-t)$, we obtain a unique (as we have just shown) curve with the nine-letter code NZSИSZNZS. Thus, the curve considered in the second case is obtained from the curve of the first case by time reversal. In particular, these curves are isometric. The uniqueness stated in the theorem is proved. To prove the existence, it only remains to verify that the minimal curve presented above indeed has a square-to-linear ratio smaller than 6. Here, we refer to computer calculations (Theorem 8, see below). □

Corollary 1. *The square-to-linear ratio of any regular diagonal Peano curve of fractal genus* 9 *is greater than or equal to* $5\frac{2}{3}$.

4. SINGULAR POINTS

A point of a Peano curve whose image belongs to a side of the image square is said to be *singular* if it is not a corner point and the curve passes through all other points on this side of the square either before or after this point. In the first case, this point is also called an *entry point*, while in the second case, an *exit point* for this side of the square.

Singular points may occur in curves of fractal genus greater than 9. Figure 10 shows the central polygonal line of the first subdivision of a Peano curve of genus 25 with a singular point on the upper side. To shorten the notation of a chain code, we will write na instead of $a+a+\ldots+a$ (n terms). Then the shortened chain code of the polygonal line in Fig. 10 is expressed as

$$\left[4i+1+(1-i)+(1+i)+1-i-1+(i-1)-(1+i)-i+3-i-3-i+3\right]. \qquad (9)$$

In addition to the central polygonal line, Fig. 10 shows the initial and final corners of fractions of the first subdivision. To identify the curve, we assume that under the similarity between any fraction and the whole curve the beginning of the fraction corresponds to the beginning of the curve and the end corresponds to the end of the curve. Then the knowledge of the beginning and end of any fraction allows one to uniquely determine the similarity between the whole image square and this fraction. Therefore, this figure uniquely specifies the curve, and the midpoint of the upper side is a singular point of this curve.

Lemma 11. *The singular points of any unit regular fractal Peano curve are rational.*

Proof. If x is a vertex of some subdivision, then it is obviously rational. Let, for some curve $p(t)$, x be a singular point (say, the entry point) that lies on the upper horizontal side of the unit square and is not a vertex (corner) of any subdivision. Then x gives rise to a unique nested sequence of fractions $F_1 \supset F_2 \supset \ldots$ containing x, where F_k is a fraction of the kth subdivision of the curve. Here, F_{k+1} is defined as the subfraction of F_k that contains x.

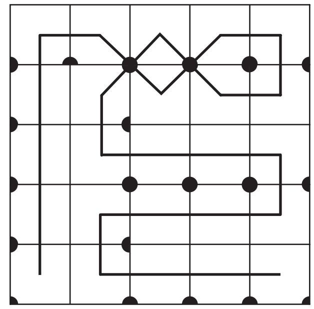

Fig. 10. A singular curve.

Since the number of pairwise nonsimilar fraction–subfraction pairs for a regular fractal curve is finite, there exists a pair of numbers $m < n$ such that the fraction F_m is similar to F_n and F_{m+1} is mapped to F_{n+1} under this similarity. In this case, this similarity maps F_{m+2} to F_{n+2} and, moreover, F_{m+k} to F_{n+k} for any k. Hence, x is a fixed point of this similarity. If we denote by a the distance from x to the upper left corner of the fraction F_m and by b the distance from the upper left corner of F_n to the upper left corner of F_m, then we obtain the equation

$$a = b + aq \qquad \text{or} \qquad a + aq = b + s, \tag{10}$$

where q is the ratio of the sizes of the fractions F_n and F_m and s is the side length of the fraction F_n. This shows that a is rational. A similar equation determines the moment of the singular point, which also turns out to be rational. □

Lemma 12. *If a diagonal curve has a singular point, then it has a diagonal junction.*

Proof. Consider a sufficiently fine subdivision of the image square such that a fraction D_k that contains a singular point (say, the entry point) lying on the upper side does not intersect the left and right sides. Let D_m and D_n be the fractions that are left and right neighbors of D_k. The curve passes through D_m and D_n after D_k. Let $D_m, D_{m+1}, \dots, D_n$ be a chain of successively traversed fractions that connects the neighbors of the fraction D_k. Let $D_1, \dots, D_k$ be a chain of fractions that connects the initial fraction with D_k. It follows from topological considerations that the central polygonal lines of the chains $D_m, D_{m+1}, \dots, D_n$ and $D_1, D_2, \dots, D_k$ intersect. The intersecting links of the central chains are the required diagonal junctions. □

Theorem 2. *A diagonal regular Peano curve with square-to-linear ratio less than* 6 *has no singular points.*

Proof. This theorem follows immediately from Lemma 12 proved above and from the earlier proved Lemma 8. □

5. RATIONALITY OF THE SQUARE-TO-LINEAR RATIO

A point of a curve whose image is a corner point of a fraction of some subdivision is called a *corner* point of a Peano curve.

Lemma 13. *Let $A = p(a)$ and $B = p(b)$ denote two points of the unit image square of a regular Peano curve $p(t)$ of genus g that have different ordinates and abscissas. If B is not a corner point of the curve $p(t)$, then there exists a corner point $B' = p(b')$ such that the square-to-linear ratio of the pair (a, b') is greater than that of the pair (a, b).*

Proof. Without loss of generality, we may assume that A lies to the left of and below B; i.e., both coordinates of B are greater than those of A.

To construct b', consider a fraction J of the nth subdivision that contains b and whose side is less than the minimum of the differences of the coordinates of the points A and B. Then J is a square that contains B. Let B' denote the upper right corner of this square, which is farthest from A. Then we take the moment of this corner in J as b'.

Next, consider the greatest $m \geq n$ for which b belongs to the same fraction of the mth subdivision as b'. In this case, we have the inequality

$$|b - b'| \leq \frac{1}{g^m}. \tag{11}$$

Since the fraction of the $(m+1)$th subdivision that contains B' does not contain B, at least one of the coordinates of B' is greater than the respective coordinate of the point B by at least $\frac{1}{g^{(m+1)/2}}$, which is the side length of the square of a fraction of the $(m+1)$th subdivision. Let A have coordinates A_1 and A_2, B have coordinates B_1 and B_2, and B' have coordinates B'_1 and B'_2, respectively. Suppose, for definiteness, that $B'_1 - B_1 \geq B'_2 - B_2$. Then

$$B'_1 - B_1 \geq \frac{1}{g^{(m+1)/2}}. \tag{12}$$

Using inequalities (11) and (12) and setting $x = \frac{1}{g^{m/2}}$, we obtain the following inequality for the square-to-linear ratio of the pair (a, b'):

$$\frac{(B'_1 - A_1)^2 + (B'_2 - A_2)^2}{b' - a} \geq \frac{(B_1 - A_1)^2 + 2x\sqrt{g^{-1}}(B_1 - A_1) + (B_2 - A_2)^2}{b - a + x^2}.$$

To prove that the square-to-linear ratio of the pair (a, b') is greater than the ratio of the pair (a, b) for sufficiently large m, it suffices the verify that the derivative of the right-hand side with respect to x is positive at $x = 0$. □

Lemma 14. *For any noninteger real number x and any natural $q > 1$, there exists a k such that the fractional part of the number xq^k is not greater than $1 - \frac{1}{q}$.*

Proof. Assuming the contrary, we find that the fractional part of x in the base-q numeral system is represented as an infinite fraction consisting of maximal digits. But such a fraction represents unity. □

Lemma 15. *Let $A = p(a)$ and $B = p(b)$ denote two points of the unit image square of a regular Peano curve $p(t)$ of genus g that have different ordinates but identical abscissas. If B does not lie on a horizontal (parallel to the abscissa axis) side of some fraction (of a certain subdivision) of the curve p, then there exists a point $B' = p(b')$, arbitrarily close to B, that has the same abscissa as B and is such that the square-to-linear ratio of the pair (a, b') is greater than that of the pair (a, b).*

Proof. Denote the ordinate of a point Z of the square by $\operatorname{Im} Z$. We will assume that B is located above A; i.e., B has a greater ordinate, $\operatorname{Im} B > \operatorname{Im} A$.

Suppose that B does not belong to the horizontal boundary of any fraction of any subdivision. By Lemma 14 applied to $\operatorname{Im} B$ (for $q = \sqrt{g}$), there exists a k such that the upper boundary of the kth-order fraction F that contains B lies at a distance $\geq g^{-(k+1)/2}$ from B. Denote by Δt_k the time interval between b and a point B' of the upper boundary of F that has the same abscissa as B. Then the ordinate of B' is greater than that of B by $\Delta y_k \geq g^{-(k+1)/2}$.

The square-to-linear ratio for a pair a, t in the case when $p(t)$ has the same abscissa as A is given by the following function of three variables:

$$f(t, x, y) = \frac{(x - y)^2}{t - a}, \tag{13}$$

where $y = \operatorname{Im} p(t)$ and $x = \operatorname{Im} A$.

In this case, the difference between the square-to-linear ratios of the pairs $(a, b+\Delta t_k)$ and (a, b) is represented as $\frac{\partial f(t,x,y)}{\partial t}\Delta t_k + \frac{\partial f(t,x,y)}{\partial y}\Delta y_k + o(\Delta y_k)$ as $k \to \infty$. Since $\Delta t_k = o(\Delta y_k)$ in this case and the derivative $\frac{\partial f(t,x,y)}{\partial y}$ is positive, it follows that $f(t, x, y)$ is not a local maximum. $\square$

Theorem 3. *The square-to-linear ratio of a regular Peano curve can attain its maximum either on a pair of corners or on a pair of singular points of some subdivision*; *in the latter case, either this pair of points has identical abscissas and both points lie on the horizontal boundaries of fractions, or the pair has identical ordinates and both points belong to the vertical boundaries of fractions.*

Proof. Let $A = p(a)$ and $B = p(b)$ with $\operatorname{Re} A \neq \operatorname{Re} B$ and $\operatorname{Im} A \neq \operatorname{Im} B$ be a pair of points of a curve $p(t)$ with the maximum square-to-linear ratio. Then the fact that both of them are corner points follows immediately from Lemma 13.

If the abscissas of the points $A = p(a)$ and $B = p(b)$ coincide, then Lemma 15 implies that these points lie on the horizontal sides of their fractions. Let, for definiteness, $a < b$. Then $p(a)$ is the exit point for its side, because for any other point of this side the squared distance to B is greater and the ratio of this squared distance to the time interval is not greater. Hence, the time interval from this point to b must be greater than $b - a$; i.e., the curve passes through this point earlier. Similar arguments show that B must be the entry point of the corresponding side.

The case of coinciding ordinates is considered in a similar way. $\square$

Corollary 2 (Bauman). *For a regular Peano curve that maps a unit interval onto a unit square, the square-to-linear ratio is a rational number.*

Proof. Theorem 3 and Lemma 11 immediately imply that the numerator of the maximum square-to-linear ratio is rational. The denominator is rational because the singular and corner moments of a regular Peano curve are rational. The rationality of the former is proved in Lemma 11, and the rationality of the latter is proved analogously. $\square$

6. DEPTH

Junctions. A junction PQ of fractions P and Q of the kth subdivision is called a *derived junction* of a junction RS of fractions R and S of a coarser subdivision if P lies in R and Q lies in S.

Lemma 16. *Any junction is similar to a derived junction of a junction of fractions of the first subdivision.*

Proof. We prove this lemma by induction on the order k of fractions in the junction. For $k = 1$, the assertion is trivial. Suppose that it is valid for the junctions of the kth subdivision. Consider a junction of fractions of the $(k + 1)$th subdivision. If this junction is not a derived junction of a junction of the first subdivision, then it is completely contained in one fraction of the first subdivision. In this case, applying the similarity transformation that maps this fraction of the first subdivision to the whole curve, we map this junction to a similar junction of fractions of the kth subdivision; thus, our assertion reduces to the induction hypothesis. $\square$

For any junction, we define its *depth* as the minimal k for which this junction is similar to a junction of a pair of fractions of the kth subdivision.

For a regular Peano curve, we define its *depth* as the maximum depth of its junctions.

Note that the depth of any curve is not less than 1. One can prove that the depth of any regular Peano curve is not greater than 9.

Theorem 4. *The depth of the minimal diagonal curve of genus* 9 *is* 1. *The depth of the first Peano curve is* 1. *The depth of the Peano–Hilbert curve is* 2.

Proof. The nine-letter code of the minimal curve is NZSИSZNZS. The junctions of the first subdivision have the codes NZ, ZS, SИ, ИS, SZ, and ZN. Note that the horizontal junctions SИ and ZN are isometric to each other and to the vertical junctions NZ and ИS.

For a fraction of type N, the code of the second subdivision is N ... S; for a fraction of type Z, the code of the second subdivision is Z ... И; for a fraction of type S, the code of the second subdivision is S ... N; and for a fraction of type И, the code of the second subdivision is И ... Z. Using this table, we find the derived junctions of all vertical junctions of the first subdivision: NZ′ = SZ, ZS′ = ИS, ИS′ = ZS, and SZ′ = NZ. Thus, we can see that all derived junctions coincide with junctions of the first subdivision. Thus, the depth of this curve is 1.

The nine-letter code of the first Peano curve is NИNИNИNИN. Hence, this curve has only two types of junctions, NИ and ИN, each having a horizontal and a vertical variants. Each of this junctions is a derived junction of itself. Therefore, the depth of the first Peano curve is 1.

We leave it to the reader to determine the depth of the Peano–Hilbert curve. □

7. PARTIAL SQUARE-TO-LINEAR RATIOS

For a pair of points $p(a) = (A_1, A_2)$, $p(b) = (B_1, B_2)$ of a Peano curve $p(t)$, we define *horizontal* and *vertical* square-to-linear ratios as

$$\frac{(A_1 - B_1)^2}{|b - a|} \qquad \text{and} \qquad \frac{(A_2 - B_2)^2}{|b - a|}, \tag{14}$$

respectively. We call each of these ratios *partial*, as opposed to the *total* ratio, which is the sum of the vertical and horizontal ratios.

The theory developed in the previous section for the total square-to-linear ratio can also be carried over to the partial ratios, essentially without changing the proofs. The main result for the partial ratios is formulated as follows.

Theorem 5. *The vertical* (*horizontal*) *square-to-linear ratio of a regular Peano curve can attain its maximum on a pair of corners or a pair of singular points of some subdivision*; *in the latter case, this pair of points lies on the horizontal* (*respectively, vertical*) *boundaries of fractions.*

A pair of points of a Peano curve is called *primary* if it bounds a segment of the curve that is not similar to any segment of this curve of greater diameter.

Lemma 17 (on a primary ratio)**.** *The following inequality holds for a primary pair of points* $A = p(x) = (A_1, A_2)$, $B = p(y) = (B_1, B_2)$ *of a regular unit Peano curve of genus* $g \ge 9$ *and depth* d:

$$\frac{|A_1 - B_1|}{|x - y|} \le g\sqrt{g^d}.$$

Proof. If a primary pair is not contained in a junction of the first subdivision, then the time interval between the points of this pair is not less than $\frac{1}{g}$, whereas the difference of their coordinates is not greater than the side of the image square (i.e., 1). Therefore, the distance-to-time ratio for these points is estimated from above by g, which is less than $g\sqrt{g^d}$.

Suppose that the points are contained in a junction of the uth subdivision and are not contained in a junction of the $(u+1)$th subdivision, so that $u \le d$ by the definition of the depth of a curve. Then, the following inequalities hold:

$$s = |A_1 - B_1| \le \frac{2}{\sqrt{g^u}}, \qquad t = |x - y| \ge \frac{1}{g^{u+1}}, \tag{15}$$

because the projection of a junction of fractions of the uth subdivision onto the abscissa axis is not greater than $\frac{2}{\sqrt{g^u}}$, while the time interval between them is not less than $\frac{1}{g^{u+1}}$.

Moreover, if the time interval between the points is not greater than $\frac{2}{g^{u+1}}$, then the pair lies in the union of a chain of three fractions of order $u+1$, whose projection is not greater than $\frac{3}{\sqrt{g^{u+1}}}$.

Therefore, the ratio s/t does not exceed $3\sqrt{g^{u+1}}$ in this case. If the time interval is greater than $\frac{2}{g^{u+1}}$, then the ratio s/t is estimated from above by $g\sqrt{g^u}$.

If $\sqrt{g} \geq 3$, then we have

$$\frac{s}{t} \leq g\sqrt{g^u} \qquad (16)$$

in both cases. □

A pair of points obtained from a primary pair by a similarity that takes the whole curve to a fraction of the first subdivision is called *secondary*.

Lemma 18. *The maximum of a partial square-to-linear ratio of a regular Peano curve of depth d is attained either on a primitive pair of points or on a secondary pair.*

Proof. Any similarity of fractions either preserves the horizontal and vertical square-to-linear ratios or interchanges them. If all fractions of the first subdivision are similar to the whole curve with the preservation of horizontal ratios, then the same is true for fractions of all subdivisions. Since any pair of points is similar to a primary pair, a pair with maximum horizontal ratio is similar in this case to a primary pair with maximum horizontal ratio.

If the similarity between some fraction of the first subdivision and the whole curve interchanges the horizontal and vertical ratios, then a primary pair of points similar to a pair of points with maximum horizontal ratio may have other horizontal ratio. In this case, applying to this primary pair a similarity transformation that maps the whole curve to a fraction of the first subdivision and interchanges the horizontal and vertical ratios, we obtain a secondary pair with maximum horizontal ratio. □

Theorem 6. *For a regular Peano curve of genus $g \geq 9$ and depth d, a partial square-to-linear ratio attains its maximum on a pair of corner or a pair of singular points of the subdivision with number $\leq d+4$. For the vertical* (*horizontal*) *ratio, the extremal pair lies on the horizontal* (*respectively, vertical*) *sides of the fractions of this subdivision.*

Proof. Let X_1 and X_2, $X_1 < X_2$, be the abscissas of a primary or secondary (Lemma 18) pair of points with maximum horizontal square-to-linear ratio, and let t_1 and t_2, $t_1 < t_2$, be the moments of these points. As follows from Theorem 5, the numbers X_2 and X_1 are $\sqrt{g}$-rational. Let k be the greatest natural number such that $X_2\sqrt{g^k}$ is not integer. Then the fractional part of this number is not greater than $1 - \frac{1}{\sqrt{g}}$.

Consider a fraction Q of the kth subdivision that contains $p(t_2)$. Denote by $p(t')$ the entry point on the right side of Q with abscissa X'. Then the following inequalities hold:

$$\Delta X = X' - X_2 \geq \frac{1}{\sqrt{g^{k+1}}}, \qquad \Delta t = t' - t_2 \leq \frac{1}{g^k}. \qquad (17)$$

Since the pair t_1, t_2 is maximal, setting $t = t_2 - t_1$, we have the inequality

$$\frac{(X_2 - X_1)^2}{t} \geq \frac{(X_2 - X_1 + \Delta X)^2}{t + \Delta t}. \qquad (18)$$

Multiplying by the denominators and canceling out equal terms, we can reduce this inequality to the following equivalent one:

$$\Delta t(X_2 - X_1)^2 \geq 2t\Delta X(X_2 - X_1) + (\Delta X)^2 t. \qquad (19)$$

This inequality implies the following:

$$\Delta t(X_2 - X_1)^2 > 2t\Delta X(X_2 - X_1), \qquad (20)$$

which is equivalent to the following:

$$\frac{\Delta X}{\Delta t} < \frac{X_2 - X_1}{2t}. \tag{21}$$

By inequalities (17), the left-hand side of (21) is greater than $\sqrt{g^{k-1}}$, whereas the right-hand side is estimated from above by $\frac{1}{2}g\sqrt{g^d}$ for a primary pair in view of Lemma 17 and by $\frac{1}{2}g\sqrt{g^{d+1}}$ for a secondary pair. Therefore, inequality (21) implies the inequality

$$\sqrt{g^{k-1}} < \frac{1}{2}g\sqrt{g^{d+1}}, \tag{22}$$

which yields $k - 1 < d + 3$, i.e., $k \le d + 3$.

As a result, we find that $X_2\sqrt{g^{d+4}}$ is an integer number. Thus, the point $p(t_2)$ belongs to the right boundary of a fraction of the $(d + 4)$th subdivision. Since the ratio is maximal, this point is the entry point for the right side of this fraction. □

8. STABILIZATION OF THE SQUARE-TO-LINEAR RATIO

Lemma 19. *The following inequality holds for a primary pair of points $X = p(x)$, $Y = p(y)$ of a regular Peano curve of genus $g \ge 9$ and depth d:*

$$\frac{|X - Y|}{|x - y|} \le g\sqrt{2g^d}.$$

Proof. If a primary pair is not contained in a junction of the first subdivision, then the distance between the points of the pair is not greater than the diagonal of the square, i.e., $\sqrt{2}$, and the time interval between these points is at least $\frac{1}{g}$. Therefore, the distance-to-time ratio for these points is estimated from above by $g\sqrt{2}$, which is less than $g\sqrt{2g^d}$.

Suppose that the points are contained in a junction of the uth subdivision and are not contained in a junction of the $(u + 1)$th subdivision, so that $u \le d$ by the definition of the depth of a curve. Then the following inequalities hold:

$$s = |X - Y| \le \frac{2\sqrt{2}}{\sqrt{g^u}}, \qquad t = |x - y| \ge \frac{1}{g^{u+1}}, \tag{23}$$

because the diameter of a junction of fractions of order u is not greater than $\frac{2\sqrt{2}}{\sqrt{g^u}}$ ($\frac{\sqrt{5}}{\sqrt{g^u}}$ for a junction along a side), and the time interval between the points is not less than the time $\frac{1}{g^{u+1}}$.

Moreover, if the time interval between the points is not greater than $\frac{2}{g^{u+1}}$, then the pair belongs to the union of a chain of three fractions of order $u + 1$, whose diameter is not greater than $\frac{3\sqrt{2}}{\sqrt{g^{u+1}}}$. Therefore, the ratio s/t is not greater than $3\sqrt{2g^{u+1}}$ in this case. If the time interval is greater than $\frac{2}{g^{u+1}}$, then the ratio s/t is estimated from above by $g\sqrt{2}\sqrt{g^u}$.

If $\sqrt{g} \ge 3$, then we have

$$\frac{s}{t} \le g\sqrt{2g^u} \tag{24}$$

in both cases. □

Lemma 20. *Let $p(t)$ be a regular Peano curve of genus $g \ge 9$ and depth d. Then a primary pair of points $p(t_1) = (X_1, Y_1)$ and $p(t_2) = (X_2, Y_2)$ with maximum total square-to-linear ratio and with $|X_1 - X_2| \ge |Y_1 - Y_2|$ belongs to the vertical boundaries of fractions of the $(d+3)$th subdivision.*

Proof. Without loss of generality, we assume that $p(t_1)$ lies below and to the left of $p(t_2)$. Thus, the differences of coordinates $X_2 - X_1$ and $Y_2 - Y_1$ are nonnegative.

In this case, we are going to prove that $X_2\sqrt{g^{d+3}}$ is an integer number. Assuming the contrary, by Lemma 14 we find a $k \geq d+3$ such that the fractional part of $X_2\sqrt{g^k}$ is not greater than $1-\frac{1}{\sqrt{g}}$.

Consider a fraction Q of the kth subdivision that contains $p(t_2)$. Denote by $p(t') = (X', Y_2)$ a point on the upper side of Q that has the same ordinate as $p(t_2)$. Then the following inequalities hold:

$$X' - X_2 \geq \frac{1}{\sqrt{g^{k+1}}}, \qquad t' - t_2 \leq \frac{1}{g^k}. \tag{25}$$

Let us show that the assumption $k \geq d+3$ contradicts the maximality of the square-to-linear ratio of the pair $p(t_1), p(t_2)$, because this ratio for the pair $p(t_1), p(t')$ turns out to be greater; i.e., let us prove the inequality

$$\frac{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}{t} < \frac{(X' - X_1)^2 + (Y_2 - Y_1)^2}{t + \Delta t} \tag{26}$$

for $t = t_2 - t_1$ and $\Delta t = t' - t_2$.

Multiplying by the denominators and canceling out equal terms, we can transform inequality (26) into an equivalent one

$$\Delta t\big((X_2 - X_1)^2 + (Y_2 - Y_1)^2\big) < 2t\Delta X(X_2 - X_1) + (\Delta X)^2 t, \tag{27}$$

where $\Delta X = X' - X_2$.

This inequality follows from the stronger inequality

$$\Delta t\big((X_2 - X_1)^2 + (Y_2 - Y_1)^2\big) \leq 2t\Delta X(X_2 - X_1), \tag{28}$$

which is equivalent to the following:

$$\frac{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}{t} \leq \frac{2\Delta X}{\Delta t}\frac{X_2 - X_1}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}. \tag{29}$$

By Lemma 19, the left-hand side of the last inequality is not greater than $g\sqrt{2g^d}$; on the right-hand side, in view of (25), the first fraction is estimated from below by $2\sqrt{g^{k-1}}$, while the second is $\geq \frac{1}{\sqrt{2}}$, because $|X_1 - X_2| \geq |Y_1 - Y_2|$. Therefore, inequality (26) is valid provided that $d + 2 \leq k - 1$. $\square$

Lemma 21. *Let Q denote the total square-to-linear ratio and Q_x be the horizontal ratio for the curve $p(t)$. Let t_1, t_2 be a pair of moments with maximum total square-to-linear ratio. Then the following inequality holds for the difference ΔY between the ordinates of the points $p(t_1)$ and $p(t_2)$:*

$$\Delta Y \geq \sqrt{t_2 - t_1}\sqrt{Q - Q_x}. \tag{30}$$

Proof. Let $p(t_1) = (X_1, Y_1)$ and $p(t_2) = (X_2, Y_2)$. Then

$$Q = \frac{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}{t_2 - t_1} \leq Q_x + \frac{(Y_2 - Y_1)^2}{t_2 - t_1},$$

which immediately implies the required inequality. $\square$

Theorem 7. *Suppose that the greatest partial square-to-linear ratio Q_x of a regular Peano curve of genus g and depth d is less than its total ratio Q. Then the (total) square-to-linear ratio attains its maximum on primary pairs of points that are corners of fractions of the subdivision with number $< d + 3 + \log_g \frac{Q^2}{4(Q-Q_x)}$.*

Proof. Let $p(t_1) = (X_1, Y_1)$ and $p(t_2) = (X_2, Y_2)$ be a primary pair of points on which the total square-to-linear ratio of the curve attains its maximum.

Without loss of generality, we will assume that the differences of coordinates X_2-X_1 and Y_2-Y_1 are nonnegative. The assumptions $t_2 > t_1$ and $Y_2 - Y_1 \leq X_2 - X_1$ do not restrict generality either. In this case $X_1\sqrt{g^{d+3}}$ and $X_2\sqrt{g^{d+3}}$ are integers by Lemma 20.

Let k be the greatest natural number such that $Y_2\sqrt{g^k}$ is not integer (such a number exists in view of Theorem 3). Then the fractional part of $Y_2\sqrt{g^k}$ is not greater than $1 - \frac{1}{\sqrt{g}}$.

Consider a fraction F of the kth subdivision that contains $p(t_2)$. Denote by $p(t') = (X_2, Y_2')$ a point on the upper side of F that has the same abscissa as $p(t_2)$. Then we have the following inequalities:

$$\Delta Y = Y_2' - Y_2 \geq \frac{1}{\sqrt{g^{k+1}}}, \qquad \Delta t = t' - t_2 \leq \frac{1}{g^k}. \tag{31}$$

Since the square-to-linear ratio of the pair t_1, t_2 is maximal, we have the inequality

$$\frac{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}{t_2 - t_1} \geq \frac{(X_2 - X_1)^2 + (Y_2 - Y_1 + \Delta Y)^2}{t_2 - t_1 + \Delta t}. \tag{32}$$

Multiplying by the denominators and canceling out equal terms, we can reduce inequality (32) to an equivalent one

$$\Delta t\big((X_2 - X_1)^2 + (Y_2 - Y_1)^2\big) \geq 2(t_2 - t_1)\Delta Y(Y_2 - Y_1) + (\Delta Y)^2(t_2 - t_1). \tag{33}$$

This inequality implies

$$\Delta t\big((X_2 - X_1)^2 + (Y_2 - Y_1)^2\big) > 2(t_2 - t_1)\Delta Y(Y_2 - Y_1), \tag{34}$$

which is equivalent to the following:

$$\frac{\Delta Y}{\Delta t} < \frac{Q}{2(Y_2 - Y_1)}. \tag{35}$$

By Lemma 21, we have $Y_2 - Y_1 \geq \sqrt{t_2 - t_1}\sqrt{Q - Q_x}$. Since $t_2 - t_1 \geq \frac{1}{g^{d+1}}$ for a primary pair, we arrive at the following inequality:

$$\frac{\Delta Y}{\Delta t} < \sqrt{g^{d+1}}\frac{Q}{2\sqrt{Q - Q_x}}. \tag{36}$$

Now, recalling inequalities (31), we conclude that

$$\sqrt{g^{k-d-2}} < \frac{Q}{2\sqrt{Q - Q_x}}. \tag{37}$$

Squaring this inequality and taking the logarithm of the result obtained to the base g, we find

$$k < d + 2 + \log_g \frac{Q^2}{4(Q - Q_x)}; \tag{38}$$

hence, recalling the definition of k, we obtain the assertion of the theorem. □

9. THE SQUARE-TO-LINEAR RATIO OF THE MINIMAL CURVE

This section is devoted to the proof of the following theorem.

Theorem 8. *The square-to-linear ratio of the minimal* N*-shaped Peano curve is* $5\frac{2}{3}$.

The proof given below is of *virtual* character. It is considerably based on computer calculations. The following lemma summarizes the necessary results of computer calculations.

Lemma 22. *The maximum of the square-to-linear ratios of pairs of corners of the fifth subdivision of the minimal* N*-shaped Peano curve is* $5\frac{2}{3}$*, while the maximum of the horizontal square-to-linear ratios of pairs of corners of the fifth subdivision of this curve is* $5\frac{1}{3}$.

Proof. The program compiled by K. Bauman calculated[2] the square-to-linear ratios for all pairs of corners of the fifth subdivision of the minimal N-shaped curve of genus 9.

This maximum value $5\frac{2}{3}$ is attained, in particular, for a pair of points with coordinates $\left(\frac{3}{3^5},0\right)$ and $\left(0,\frac{5}{3^5}\right)$. The moments of these points are $\frac{20}{177\,147}$ and $\frac{38}{177\,147}$, respectively, the squared distance between these points is $\frac{4}{6561}$, and the time interval between them is $\frac{2}{19\,683}$.

The calculations also showed that the horizontal square-to-linear ratio Q_x attains its maximum, in particular, on a pair of points with coordinates $\left(0,\frac{9}{3^5}\right)$ and $\left(\frac{4}{3^5},\frac{9}{3^5}\right)$. The moments of these points are $\frac{74}{177\,147}$ and $\frac{83}{177\,147}$, respectively, the squared distance between these points is $\frac{16}{9^5}$, and the time interval between them is $\frac{9}{177\,147}=\frac{1}{19\,683}$. The horizontal square-to-linear ratio for these points is $Q_x=5\frac{1}{3}$. □

Lemma 23. *The maximum horizontal and vertical square-to-linear ratios of the minimal* N*-shaped Peano curve are equal.*

Proof. Since the second fraction of the first subdivision of the N-shaped curve is coded by the letter Z of horizontal type, this fraction is homothetic to the whole curve turned through an angle of 90°. Therefore, the maximum vertical square-to-linear ratio Q'_y of this fraction coincides with the maximum horizontal ratio Q_x of the whole curve, while the maximum horizontal ratio Q'_x of this fraction coincides with the maximum vertical ratio Q_y of the whole curve. On the other hand, the maximum partial ratio of a fraction does not exceed the corresponding maximum ratio of the whole curve. Thus, the inequalities $Q_y=Q'_x\le Q_x$ and $Q_x=Q'_y\le Q_y$ hold, which imply $Q_x=Q_y$. □

Let us denote by $|p|_n$ the maximum of the square-to-linear ratios of a regular Peano curve $p(t)$ over all possible pairs of corner points of its nth subdivision. We will refer to $|p|_n$ as the *n-corner* ratio. By $|p|$ we denote the square-to-linear ratio of the curve $p(t)$.

Lemma 24. *Let $p(t)$ be a regular Peano curve of genus g and depth d. Then the following inequality holds for any $n\ge d$:*

$$|p|_n\le|p|\le|p|_n\left(1+\frac{2}{g^{(n-d-1)}}\right).$$

Proof. Let $t,t'\in[0,1]$. The square-to-linear ratio of p for this pair is the same as that for a similar primary pair. Therefore, we may assume that t and t' do not belong to adjacent fractal periods of the $(d+1)$th subdivision. Hence,

$$|t-t'|\ge\frac{1}{g^{1+d}}. \tag{39}$$

Consider the fractions of the curve of the nth subdivision that contain $p(t)$ and $p(t')$. The images of these fractions are squares that contain the points $p(t)$ and $p(t')$, respectively. Therefore, the distance between $p(t)$ and $p(t')$ is not greater than the distance between the farthest corners of these squares. Denote by τ and τ' the corresponding corner moments. Then

$$|p(t)-p(t')|\le|p(\tau)-p(\tau')|,\qquad|t-\tau|\le\frac{1}{g^n},\qquad|t'-\tau'|\le\frac{1}{g^n}. \tag{40}$$

[2]The calculations used integer type data and so are exact.

Inequalities (39) and (40) allow us to estimate from above the square-to-linear ratio of the pair $p(t), p(t')$ in the following way:

$$\frac{|p(t)-p(t')|^2}{|t-t'|} \le \frac{|p(\tau)-p(\tau')|^2}{|t-t'|} = \frac{|p(\tau)-p(\tau')|^2}{|\tau-\tau'|}\frac{|\tau-\tau'|}{|t-t'|}$$

$$\le |p|_n\left(1+\frac{|t-\tau|+|t'-\tau'|}{|t-t'|}\right) \le |p|_n\big(1+2g^{1+d-n}\big). \quad \square \qquad (41)$$

Proof of Theorem 8. Since the depth of the minimal N-shaped Peano curve is 1 (Theorem 4) and it has no singular points (by Lemma 12), Theorem 6 implies that the horizontal square-to-linear ratio of this curve attains its maximum on a certain pair of corners of fractions of the fifth subdivision and is equal to $Q_x = 5\frac{1}{3}$ according to Lemma 22.

Next, an upper estimate for the total square-to-linear ratio $Q < 5\frac{2}{3}\big(1+\frac{2}{9^3}\big) < 6$ can be obtained by Lemmas 24 and 22. Since $Q \ge 5\frac{2}{3}$, we obtain the following inequalities:

$$\log_9 \frac{Q^2}{4(Q-Q_x)} < \log_9 \frac{6^2}{4\times\frac{1}{3}} = \frac{3}{2},$$

which, in view of Theorem 7 (applicable due to Lemma 23), imply that the total square-to-linear ratio attains its maximum on pairs of corners of the fifth subdivision of the minimal Peano curve. $\square$

10. CONCLUDING REMARKS

The literature mainly deals with "square" Peano curves (i.e., curves that map an interval onto a square). Is it possible to construct Peano curves with smaller square-to-linear ratio if we remove any constraints on the form of the image? What is the form of the image for Peano curves with the minimal square-to-linear ratio? These questions are closely related to the following open problem (cf. [2]).

Problem 1. What is the minimum number κ for which there exists a continuous mapping of a unit interval onto a plane set of unit area with the maximum square-to-linear ratio equal to κ?

The minimum number κ mentioned in this problem is called the *Peano constant of free form*. If we impose a constraint on the form of the image in this problem (square, triangle, disk, etc.), then the corresponding minimum is called the *Peano constant* of this form (square, triangular, circular, etc.). For instance, all the results of the paper [2] are devoted to estimates for the square Peano constant.

Stabilization problem. The theory developed in this paper leads to the following natural conjecture: for a regular Peano curve of depth d, the square-to-linear ratio attains its maximum either on a pair of corners of fractions of the $(d+4)$th subdivision or on a pair of singular points of fractions of the $(d+4)$th subdivision.

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